Review

(I) Newton's Method: Systems of non-linear equation

Want to solve $F(X)=0$ where $X \in \mathbb{R}^n$

$F: \mathbb{R}^n \to \mathbb{R}^n$

(system of n-equations, n unknowns)

$(k+1)$'th iteration

$$X^{(k+1)} = X^{(k)} + H^{(k)}$$

where $F'(X^{(k)})\, H^{(k)} = -F(X^{(k)})$

Jacobian: $F'(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$

e.g. Perform two iterations of Newton's method on

$$\begin{cases} xy^2 + x^2y + x^4 = 3 \\ x^3y^5 - 2x^5y - x^2 = -2 \end{cases}$$

starting with $(1,1)$

Soln: $f_1(x,y) = xy^2 + x^2y + x^4 - 3$

$f_2(x,y) = x^3y^5 - 2x^5y - x^2 = -2$

$$F'(x,y) = \begin{bmatrix} y^2 + 2xy + 4x^3 & 2xy + x^2 \\ 3x^2y^5 - 10x^4y - 2x & 5x^3y^4 - 2x^5 \end{bmatrix}$$

$$\Rightarrow F'(1,1) = \begin{bmatrix} 7 & 3 \\ -9 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} X^{(1)} = X^{(0)} + H^{(0)} \\ F'(1,1)H^{(0)} = -F(1,1) \end{cases}$$

solve for $H^{(0)}$, plug it in ...

— X —

(II) Numerical Differentiation

1. Using the Taylor series for $f(x+h)$ & $f(x-h)$

$$f'(x) = \underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{\text{approx}} - \underbrace{\frac{h^2}{6} f'''(\xi)}_{\text{error}}$$

→ error is $O(h^2)$

you should know how to derive expressions like this

$$f''(x) = \underbrace{\frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right]}_{\text{approx}} - \underbrace{\frac{h^2}{12} f^{(4)}(\xi)}_{\text{error}}$$

error is $O(h^2)$

Using Polynomial interpolation to approx. the deriv. at x_j

$$f(x) = \sum_{i=0}^{n} f(x_i) L_i(x) + \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

Lagrange form of interpolating polynomial

$$\Rightarrow f'(x_j) = \sum_{i=0}^{n} f(x_i) L_i'(x_j) + \frac{1}{(n+1)!} f^{(n+1)}(\xi_{x_j}) \prod_{i \neq j} (x_j - x_i)$$

Richardson Extrapolation: (for approx. $f'(x)$)

(I) Select h, compute $D(n,0) = \varphi(h/2^n)$

where $n = 0, \ldots, M$

$$\varphi(h) = \frac{f(x+h) - f(x-h)}{2h}$$

(II) Compute $D(n,k) = \frac{4^k}{4^k - 1} D(n, k-1) - \frac{1}{4^k - 1} D(n-1, k-1)$

where $k = 1, \ldots, M$ & $n = k, k+1, \ldots, M$

Theorem: Suppose $f'(x)$ $L = \varphi(h) + \sum_{j=1}^{\infty} a_{2j} h^{2j}$

Then $D(n,k-1) = L + \sum_{j=k}^{\infty} A_{j,k} \left(\frac{h}{2^n}\right)^{2j}$

approx error: $O(h^{2k})$

(III) Numerical Integration

$$f(x) \approx p(x) = \sum_{i=0}^{n} f(x_i)\, \ell_i(x)$$

Lagrange form of interpolating polynomial

Integrate both sides

$$\Rightarrow \cdots \Rightarrow \int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

where $A_i = \int_a^b \ell_i(x)\,dx$

Examples: Trapezoid rule: $A_0 = A_1 = \frac{b-a}{2}$

Simpson's rule: $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$

$$A_0 = \frac{b-a}{6} = A_2, \quad A_1 = \frac{4}{6}(b-a)$$

Method of undetermined coefs to find the A_i's

If you have $n+1$ nodes, define $f_i(x) = x^i$ $i = 0, \ldots, n-1$

then $\int_a^b f_i(x)\,dx = \sum_{j=0}^{n} A_j f_i(x_j)$ (= exactly) → $n+1$ equations & unknowns, solve to get A_i's

Theorem: $\int_a^b F(x)\,dx - \sum_{i=0}^{n} A_i F(x_i) \leq \frac{M_n}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x - x_i)\,dx$

where $M_n \geq \max_{x \in [a,b]} |F^{(n+1)}(x)|$

when $x_i = \cos\frac{(i+1)\pi}{n+2}$ (roots of Chebyshev poly of the 2nd kind)

$U_{n+1}(x)$

$= \frac{\sin((n+2)\cos^{-1}x)}{\sin(\cos^{-1}x)}$

then

$$\left| \int_{-1}^{1} F(x)\,dx - \sum_{i=0}^{n} A_i F(x_i) \right| \leq \frac{M_n}{(n+1)!} 2^{-n}$$